in hand. A more precise presentation of the idea of limit than is customary would greatly facilitate the use of the notion in college teaching.

Recent numbers of the MONTHLY have contained articles by Mr. Cheney [*1920*, 53] and Professor Lovitt [*1920*, 465] on geometric proofs of the law of tangents. A proof distinct from those as yet proposed is given in the last discussion this month by Professor Epperson.

I. ON EXACT DIFFERENTIALS.

By J. W. CAMPBELL, University of Alberta.

The criterion usually given that the differential $Pdx + Qdy$ shall be exact is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The functions P and Q are therefore assumed to be differentiable, and they are also assumed to have the other properties necessary for the application of Green's theorem. The purpose of this note is to suggest, in this case and in the case of n variables, an integral condition in which P and Q do not necessarily satisfy the hypotheses of Green's theorem.

THEOREM I.[1] *The necessary and sufficient conditions that*

$$Pdx + Qdy \tag{1}$$

shall be an exact differential are that P and Q shall be integrable with regard to x and y, respectively, and that

$$\int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy \equiv \int_{y_0}^{y} Q(x, y)dy + \int_{x_0}^{x} P(x, y_0)dx, \tag{2}$$

where (x_0, y_0) is an arbitrary fixed point in the vicinity of which P and Q are integrable.

For if (1) is exact it must be of the form

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy,$$

whence

$$\frac{\partial u}{\partial x} = P, \qquad \frac{\partial u}{\partial y} = Q.$$

Therefore

$$\begin{aligned} u &= \int_{x_0}^{x} P(x, y)dx + f(y) \\ &= \int_{y_0}^{y} Q(x, y)dy + g(x), \end{aligned} \tag{3}$$

where f and g are arbitrary functions of integration.

[1] It is assumed that the region under consideration is rectangular.—EDITOR.

The two expressions (3) for u must be identically equal in x and y, and therefore

$$
\begin{aligned}
& f(y) \equiv \int_{y_0}^{y} Q(x_0, y)dy + g(x_0), \\
(4) \qquad & g(x) \equiv \int_{x_0}^{x} P(x, y_0)dx + f(y_0), \\
& f(y_0) = g(x_0).
\end{aligned}
$$

The substitution of (4) in (3) gives (2), and therefore (2) is necessary.

It is also sufficient. For if $u(x, y)$ represents the common value of the two expressions in (2), then

$$P = \frac{\partial u}{\partial x}, \qquad Q = \frac{\partial u}{\partial y}$$

and therefore (1) is exact.

THEOREM II.[1] *The necessary and sufficient conditions that*

$$\sum_{i=1}^{n} X_i dx_i \tag{5}$$

shall be an exact differential are that the functions X_i shall be integrable with respect to x_i and that

$$\sum_{i=1}^{n} \int_{x_i^{(0)}}^{x_i} X_i dx_i \tag{6}$$

shall be invariant under any interchange of subscripts, where in the jth term of each sum so obtained the x_i with respect to which integrations have been made in the first $(j-1)$ terms are replaced by $x_i^{(0)}$, $(j = 2, \cdots, n)$.

For if (5) is exact it must be of the form

$$\sum_{i=1}^{n} \frac{\partial U}{\partial x_i} dx_i$$

and therefore

$$X_i = \frac{\partial U}{\partial x_i}, \qquad (i = 1, \cdots, n).$$

Therefore

$$U = \int_{x_i^{(0)}}^{x_i} X_i dx_i + Y_i, \qquad (i = 1, \cdots, n) \tag{7}$$

where Y_i is an arbitrary function of all the x_j except x_i.

Now let us suppose that the equations (7) imply the stated condition in the case of $n - 1$. Then on replacing the x_i successively by $x_i^{(0)}$ in (7) we readily show that the condition is implied in the case of n. And since the implication has been proved for $n = 2$, it follows by mathematical induction that equations (7) imply the necessity of the condition as stated.

[1] It is assumed that the region is a generalized rectangle, that is, that $a_i \leqq x_i \leqq b_i$.—EDITOR.

The proof of the sufficiency is similar to that for the case $n = 2$.

To these two theorems may be added a third related theorem.

THEOREM III. *The necessary and sufficient condition that*

$$\int_{(C)} Pdx + Qdy \tag{8}$$

shall vanish, where C is any closed contour in a region in which P and Q are continuous, is that $Pdx + Qdy$ shall be an exact differential.

For if the line integral is zero and C is arbitrary, then

$$\int_{x_0, y_0}^{x, y} Pdx + Qdy$$

is a function of x and y only, and does not depend on the path.

That is,

$$\int_{x_0, y_0}^{x, y} Pdx + Qdy = v(x, y).$$

Therefore

$$P = \frac{\partial v}{\partial x}, \qquad Q = \frac{\partial v}{\partial y}$$

and the diffential is exact.

And again if $Pdx + Qdy$ is exact, it must be of the form du where u is the common value of the two expressions in (2). But the u as there defined is continuous in x and y, and therefore the total algebraic variation about a closed contour is zero.

The form of the integral expressions appearing in (2) and (6) is usually given as the formula for the integral when the differential is exact, but so far as I have been able to find the invariance of these expressions under cyclic interchange of notation has not been given as a criterion for exactness.

II. THE TEACHING OF LIMITS IN THE HIGH SCHOOL.[1]

By J. V. McKELVEY, Iowa State College.

The title of the present paper is to some extent either misleading or noncommittal. To make our purpose somewhat clearer, it may be stated that we hold no brief either for or against the teaching of limits in preparatory schools. We intend, rather, to state the results of several years' observation of high-school students during their early years in college particularly in regard to their understanding of limiting operations in the most elementary sense of the word. We open this discussion with whatever apologies may be necessary for saying some things that, perhaps, everybody knows.

To plunge rather abruptly into the midst of the question, we note that

[1] Read before the Iowa Academy of Science, April 24, 1920.